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On measurable Hermitian indefinite functions with a finite number of negative squares

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Dedicated to Professor Béla Szőkefalvi-Nagy on his 70th birthday

1. Introduction and main result

1. Let $\varkappa$ be a nonnegative integer and $0<a\leqq\infty$. We denote by $\mathfrak{P}_{\varkappa;a}$ the set of all complex functions f defined on the open interval $(-2a, 2a)$ with the following properties:

(i) $f(t)=\overline{f(-t)}$ $(-2a<t<2a)$;

(ii) the (Hermitian) kernel H_f defined by

$$H_f(t, s):=f(t-s) \quad (-a<s, t<a)$$

has $\varkappa$ negative squares;

(iii$_c$) f is continuous on $(-2a, 2a)$.

We denote by $\mathfrak{P}^m_{\varkappa;a}$ the set of all complex functions f on $(-2a, 2a)$ satisfying (i), (ii) and

(iii$_m$) f is measurable and locally bounded on $(-2a, 2a)$.

The aim of this note is to prove the following

Theorem. *The function $f\in\mathfrak{P}^m_{\varkappa;a}$ admits a unique decomposition*

$$f(t)=f_c(t)+f_s(t) \quad (-2a<t<2a) \tag{1}$$

such that $f_c\in\mathfrak{P}_{\varkappa;a}$, $f_s\in\mathfrak{P}^m_{0;a}$ and $f_s(t)=0$ a.e. on $(-2a, 2a)$.

For $\varkappa=0$ this theorem was proved by M. G. KREĬN [1], see also [2]. In this case it implies the classical result of F. RIESZ [3] stating that an arbitrary function $f\in\mathfrak{P}^m_{0;\infty}$ coincides almost everywhere with some $f_c\in\mathfrak{P}_{0;\infty}$. In connection with the paper [1] M. G. Krein asked for a generalization of his result to functions with $\varkappa$ negative squares. The Theorem above gives an affirmative answer to this question.

The proof of the Theorem will be given in section 4. Also for $\varkappa=0$ it is different from the proof of the corresponding theorem in [1]. As a main tool we use a result about the continuation of generalized functions with $\varkappa$ negative squares from a bounded interval to the whole real axis, which is perhaps of some interest in its own (see 3). In particular, it seems to be new even for $\varkappa=0$. Then it extends a classical result of M. G. Krein to positive definite generalized functions.

We mention that the decomposition of a positive definite measurable function f on R^n into the sum of a continuous positive definite function f_c and a positive definite function f_s on R^n, which vanishes almost everywhere, was proved by M. M. Crum [12], the corresponding fact for measurable positive definite functions on a locally compact group was proved by J. von Neumann and I. E. Segal, see [13].

A survey and a bibliography about positive definite functions and their generalizations can be found in [4]; continuous functions with a finite number of negative squares were considered, e.g., in [5] and [6, Parts I, IV].

2. In this section the function f is supposed to satisfy the conditions (i) and (ii). If $\varkappa=0$, then an arbitrary (even nonmeasurable) function f of this kind is bounded:

$$|f(t)| \leqq f(0) \quad (-2a < t < 2a).$$

If $\varkappa>0$ and f is continuous, then it may be unbounded at infinity (in case $a=\infty$). This holds, e.g., for $\varkappa=1$ if f has a representation of parabolic or hyperbolic type see [6, Part IV]. If f is not measurable and $\varkappa>0$, then it may be unbounded at zero. To see this we choose a (nonmeasurable) solution α of the functional equation $\alpha(t+s)=\alpha(t)\cdot\alpha(s)$, $\alpha(0)=1$, which is not locally bounded, and consider the following function f:

$$f(t) := \gamma\alpha(t)+\bar{\gamma}\overline{\alpha(t)}^{-1} \quad (-2a < t < 2a)$$

for arbitrary $0<a\leqq\infty$. The relation

$$\sum_{i,j=1}^{n} f(t_i-t_j)\xi_i\overline{\xi_j} = \gamma\sum_{i=1}^{n}\alpha(t_i)\xi_i\sum_{j=1}^{n}\alpha(t_j)^{-1}\overline{\xi_j}+\bar{\gamma}\sum_{i=1}^{n}\overline{\alpha(t_i)}\overline{\xi_i}\sum_{j=1}^{n}\overline{\alpha(t_j)}^{-1}\xi_j$$

shows that the kernel H_f has one negative and one positive square.

However, it is an open question if a measurable function f satisfying (i) and (ii) can be unbounded on some compact subinterval of $(-2a, 2a)$. Thus, we do not know whether the boundedness condition in (iii$_m$) should be imposed.

2. $\pi_\varkappa$-spaces associated with elements of $\mathfrak{P}^m_{\varkappa, 2a}$

1. Let f be a complex function on $(-2a, 2a)$ satisfying the conditions (i) and (ii). We associate with f a $\pi_\varkappa$-space $\Pi_\varkappa(f)$ as follows. Consider the linear set $\mathscr{L}_0$ of all complex functions $u: s\to u(s)$ on $(-a, a)$ that are different from zero only in a finite number of points s, and equip $\mathscr{L}_0$ with the scalar product

$$[u, v] := \sum_{-a<s,t<a} f(t-s)u(s)\overline{v(t)} \quad (u, v\in\mathscr{L}_0).$$

The conditions (i) and (ii) imply that this scalar product is Hermitian and has $\varkappa$ negative squares on $\mathscr{L}_0$. Thus $\mathscr{L}_0$ can be canonically embedded into a $\pi_\varkappa$-space, which we shall denote by $\Pi_\varkappa(f)$. The element of $\Pi_\varkappa(f)$ corresponding to a function $u\in\mathscr{L}_0$ will also be denoted by u. Moreover, we introduce the functions $\varepsilon_t\in\mathscr{L}_0$, $-a<t<a$, as follows:

$$\varepsilon_t(s):=\begin{cases}1 & \text{if } s=t\\ 0 & \text{if } s\neq t\end{cases}\quad(-a<s<a).$$

Evidently, the elements ε_t, $-a<t<a$, generate the space $\Pi_\varkappa(f)$ and we have $[\varepsilon_s,\varepsilon_t]=f(t-s)$ $(-a<s,\ t<a)$.

Let $u_1,\dots,u_\varkappa$ be elements of $\mathscr{L}_0$ such that

$$[u_j,u_k]=-\delta_{jk},\quad j,k=1,2,\dots,\varkappa.$$

We consider the Hilbert norm

$$\begin{aligned}\|x\|^2:=\sum_{j=1}^{\varkappa}|[x,u_j]|^2+\Big[x+\sum_{j=1}^{\varkappa}[x,u_j]u_j,\ x+\sum_{j=1}^{\varkappa}[x,u_j]u_j\Big]=\\ =[x,x]+2\sum_{j=1}^{\varkappa}|[x,u_j]|^2\quad(x\in\Pi_\varkappa(f))\end{aligned}\tag{2}$$

on $\Pi_\varkappa(f)$.

Lemma 1. *If the function f satisfies* (i) *and* (ii), *and is locally bounded on* $(-2a,2a)$, *then the function $t\to\|\varepsilon_t\|$ $(-a<t<a)$ is locally bounded.*

Indeed, we have from (2)

$$\|\varepsilon_s\|^2=[\varepsilon_s,\varepsilon_s]+2\sum_{j=1}^{\varkappa}|[\varepsilon_s,u_j]|^2=f(0)+2\sum_t\sum_{j=1}^{\varkappa}f(t-s)\overline{u_j(t)},$$

and the statement follows from the local boundedness of f on $(-2a,2a)$ if we observe that both summations on the right hand side are finite.

2. Now let $f\in\mathfrak{P}^m_{\varkappa;a}$. Then, besides $\Pi_\varkappa(f)$, a space $\Pi^c(f)$ can be defined as follows. Let C_a be the linear set of all continuous complex functions φ on $(-a,a)$ which vanish outside of some compact subinterval of $(-a,a)$. We define a scalar product on C_a by the formula

$$[\varphi,\psi]_C:=\int_{-a}^{a}\int_{-a}^{a}f(t-s)\varphi(s)\overline{\psi(t)}\,ds\,dt\quad(\varphi,\psi\in C_a).\tag{3}$$

It will be shown in this section that the factor space of C_a modulo the isotropic subspace $\mathscr{L}_C^0$ of C_a with respect to the scalar product $[\cdot,\cdot]_C$ can be identified with some linear manifold in $\Pi_\varkappa(f)$.

To this end, for a given $\varphi \in C_a$ we define a linear functional F_φ on $\mathscr{L}_0$ by

$$F_\varphi(u) := \sum_s u(s) \int_{-a}^{a} f(t-s)\overline{\varphi(t)}\, dt \quad (u \in \mathscr{L}_0).$$

Let $(u_n) \subset \mathscr{L}_0$ be a sequence which converges to the zero element of $\Pi_\varkappa(f)$ if $n \to \infty$. Then we have

$$\left|\sum_s u_n(s) f(t-s)\right| = |[u_n, \varepsilon_t]| \leqq \|u_n\| \, \|\varepsilon_t\|.$$

The right hand side in this relation tends to zero if $n \to \infty$, and according to Lemma 1 this convergence holds locally uniformly with respect to $t \in (-a, a)$. This implies $F_\varphi(u_n) \to 0$ $(n \to \infty)$. Therefore F_φ is continuous and can be extended by continuity to all of $\Pi_\varkappa(f)$. Hence there exists an element $\varphi \in \Pi_\varkappa(f)$ such that

$$F_\varphi(x) = [x, \varphi] \quad (x \in \Pi_\varkappa(f)).$$

In particular,

$$(4) \quad [\varepsilon_s, \varphi] = \int_{-a}^{a} f(t-s)\overline{\varphi(t)}\, dt, \; [u, \varphi] = \int_{-a}^{a} [u, \varepsilon_t]\overline{\varphi(t)}\, dt \quad (s \in (-a, a),\; u \in \mathscr{L}_0).$$

Next we show that the scalar product of two such elements $\varphi, \psi \in \Pi_\varkappa(f)$ coincides with (3). Indeed, if $\varphi \in C_a$, then there exists a sequence $(\varphi_n) \subset \mathscr{L}_0$ such that $\varphi_n \to \varphi$ in $\Pi_\varkappa(f)$. This implies that

$$[\varphi_n, \psi] \to [\varphi, \psi] \quad (n \to \infty), \quad [\varphi_n, \varepsilon_t] \to [\varphi, \varepsilon_t] \quad (n \to \infty),$$

and the latter convergence holds locally uniformly with respect to $t \in (-a, a)$. Thus we get

$$[\varphi, \psi] = \lim_{n \to \infty} [\varphi_n, \psi] = \lim_{n \to \infty} \int_{-a}^{a} [\varphi_n, \varepsilon_t]\overline{\psi(t)}\, dt =$$

$$= \int_{-a}^{a} [\varphi, \varepsilon_t]\overline{\psi(t)}\, dt = \int_{-a}^{a}\int_{-a}^{a} f(t-s)\varphi(s)\overline{\psi(t)}\, ds\, dt,$$

that is

$$[\varphi, \psi] = [\varphi, \psi]_C.$$

The factor space $C_a/\mathscr{L}_C^0$ can be identified with some linear manifold in $\Pi_\varkappa(f)$. In particular, the scalar product (3) has only a finite number $\varkappa'$, $0 \leqq \varkappa' \leqq \varkappa$, of negative squares. The completion of $C_a/\mathscr{L}_C^0$ will be denoted by $\Pi^C(f)$. It is a $\pi_{\varkappa'}$-space for some $0 \leqq \varkappa' \leqq \varkappa$, and can be identified with some (non-degenerate) subspace of $\Pi_\varkappa(f)$. Later we shall see that actually $\varkappa' = \varkappa$.

Remark 1. Instead of C_a we could have started from the space K_a of those elements of C_a which have derivatives of arbitrary order. If we again define the scalar product $[\varphi, \psi]_C$ for $\varphi, \psi \in K_a$ by the relation (3); it is easy to see that the completion of the factor space $K_a/\mathcal{L}_K^0$ coincides with $\Pi^C(f)$; here $\mathcal{L}_K^0$ denotes the isotropic subspace of K_a with respect to the scalar product $[\cdot, \cdot]_C$.

Remark 2. If the function f is continuous, that is $f\in\mathfrak{P}_{\varkappa;a}$, then the spaces $\Pi_\varkappa(f)$ and $\Pi^C(f)$ coincide. Indeed, if $s\in(-a, a)$, let $(\delta_s^{(n)})$, $n=1, 2, \ldots$, be a δ_s-sequence of elements of C_a. Then it is easy to see that $\delta_s^{(n)}\to\varepsilon_s$ if $n\to\infty$, and the inclusion $\Pi_\varkappa(f)\subset\Pi^C(f)$ follows. Thus the spaces $\Pi_\varkappa(f)$ and $\Pi^C(f)$ are identical. Obviously, in this case the space $\Pi_\varkappa(f)$ is separable.

3. Generalized functions with $\varkappa$ negative squares

1. We denote by $\mathfrak{P}^d_{\varkappa;a}$ the set of all generalized functions F on $(-2a, 2a)$ over the space K_{2a} with the following properties:

(i′) $(F, \varphi)=\overline{(F, \varphi^*)}$ $(\varphi\in K_{2a};\ \varphi^*(t):=\overline{\varphi(-t)})$;

(ii′) the kernel H_F on $K_a\times K_a$ defined by

$$H_F(\varphi, \psi):= (F, \varphi^\circ * \bar{\psi}) \quad (\varphi, \psi\in K_a,\ \varphi^\circ(t):=\varphi(-t))$$

has $\varkappa$ negative squares.

The generalized function $F\in\mathfrak{P}^d_{\varkappa;a}$ induces a scalar product

$$[\varphi, \psi]_K:= (F, \varphi^\circ * \bar{\psi}) \quad (\varphi, \psi\in K_a) \tag{5}$$

on K_a with $\varkappa$ negative squares. The corresponding $\pi_\varkappa$-space will be denoted by $\Pi_\varkappa^K(F)$.

Recall ([7, § 4]) that a family (T_t), $0\leqq t<\infty$, of bounded linear operators in a Banach space $\mathcal{B}$ is a *generalized semigroup,* if it has the following properties:

(a) $\mathfrak{D}(T_t)=:\mathfrak{D}_t$ is a closed subspace of $\mathcal{B}$ and we have

$$\mathfrak{D}_t\subset\mathfrak{D}_{t'} \quad \text{if} \quad 0\leqq t'\leqq t, \quad \overline{\bigcup_{t>0}\mathfrak{D}_t}=\mathcal{B};$$

(b) $T_0=I$, $T_{t+t'}=T_tT_{t'}$ $(t, t'\geqq 0)$;

(c) if $t_0>0$, $x\in\mathfrak{D}_{t_0}$ and $0\leqq t, t'\leqq t_0$ then $\lim\limits_{t'\to t} T_{t'}x=T_tx$.

The *infinitesimal generator* A_0 of the generalized semigroup (T_t), $0\leqq t<\infty$, is defined as follows: $\mathfrak{D}(A_0)$ is the set of all $x\in\bigcup\limits_{t>0}\mathfrak{D}_t$ such that the limit $\lim\limits_{h\downarrow 0}\frac{1}{ih}(T_hx-x)$ exists and

$$A_0x:= \lim_{h\downarrow 0}\frac{1}{ih}(T_hx-x) \quad (x\in\mathfrak{D}(A_0)),$$

see [7, § 4].

If $t\in R^1$, we denote by V_t the shift operator in K_a: $\varphi\in\mathfrak{D}(V_t)$ if either $\varphi=0$, or $\varphi\in K_a$ and $t+\operatorname{supp}\varphi\subset(-a, a)$, and if e.g., $t>0$,

$$(V_t\varphi)(s):= \begin{cases}\varphi(s-t) & \text{if} \quad -a+t\leqq s<a\\ 0 & \text{if} \quad -a<s\leqq -a+t\end{cases} \quad (\varphi\in\mathfrak{D}(V_t)).$$

It is easy to see that the operators V_t preserve the scalar product (5) on K_a. If $|t|$ is sufficiently small then $\vartheta(V_t)$ contains a $\varkappa$-dimensional negative subspace (with respect to (5)). Therefore, for these t the operator V_t is continuous in the norm topology of $\Pi_\varkappa^K(F)$ ([8, IX. 3]). The relation $V_{t+t'}=V_tV_{t'}$, $\operatorname{sgn} t=\operatorname{sgn} t'$, implies that all the operators V_t, $t\in R^1$, are continuous. Thus they can be extended by continuity to the closure $\overline{\vartheta(V_t)}$ in $\Pi_\varkappa^K(F)$. As a result we get a generalized semigroup of bounded isometric operators in $\Pi_\varkappa^K(F)$, which will also be denoted by (V_t), $0\leqq t<\infty$. Its infinitesimal generator is the operator

$$A_0 = i\frac{d}{dt}.$$

Evidently, $K_a\subset\vartheta(A_0)$ and $A_0K_a\subset K_a$. Moreover, the operator A_0 in $\Pi_\varkappa^K(F)$ is π-Hermitian (this either follows easily from the fact that the operators V_t are π-isometric, or can be checked directly). As it is real with respect to the involution $\varphi\to\varphi^*$ in K_a, its defect numbers are equal, and it is not hard to show (cf. [6, Part IV; §2]) that they are either $=0$ or $=1$.

Moreover, if $a=\infty$, then the operators V_t, $t\in R^1$, form a group of π-unitary operators. In this case the operator A_0 is π-self-adjoint.

2. Proposition 1. *If $F\in\mathfrak{P}^d_{\varkappa;\,a}$, there exists at least one generalized function $\tilde F\subset\mathfrak{P}^d_{\varkappa;\,\infty}$ which extends F to the whole real axis.*

Proof. We consider the operator A_0 in $\Pi_\varkappa^K(F)$. It admits at least one π-self-adjoint extension $\tilde A$ in $\Pi_\varkappa^K(F)$. Denote by $(\tilde U_t)$, $t\in R^1$, the group of π-unitary operators in $\Pi_\varkappa^K(F)$ generated by $\tilde A$, that is, $\tilde U_t:=\exp(it\tilde A)$ $(t\in R^1)$. Then, if $\varphi\in K_a$, we have $(\tilde U_t\varphi)(s)=\varphi(s-t)$ for sufficiently small $|t|$. In fact, $\tilde U_t$ is a π-unitary extension of the operator V_t.

An extension $\tilde F$ of F can now be defined as follows. Let $\varphi,\psi\in K_\infty$ be such that their supports are contained in closed intervals of length $<2a$, say,

$$\operatorname{supp}\varphi\subset(t'-a,\,t'+a),\quad \operatorname{supp}\psi\subset(t''-a,\,t''+a) \tag{6}$$

for some $t', t''\in R^1$. Then, if $\tilde V_t$ denotes the shift operator in K_∞ defined by

$$(\tilde V_t\varphi)(s):=\varphi(s-t)\quad(\varphi\in K_\infty,\ s,\,t\in R^1),$$

we have $\tilde V_{-t'}\varphi$, $\tilde V_{-t''}\psi\in K_a$. The scalar product $[\cdot,\cdot]_K$ can be extended by the relation

$$[\varphi,\psi]:=[\tilde U_{t'}\tilde V_{-t'}\varphi,\ \tilde U_{t''}\tilde V_{-t''}\psi]_K.$$

It is not hard to see that this definition is correct, that is, on K_a it gives the scalar product already defined, and it is independent of the choice of t' and t'' if only the translations $\tilde V_{-t'}$ and $\tilde V_{-t''}$ map φ and ψ, respectively, into K_a.

If $\varphi, \psi \in K_\infty$ do not satisfy conditions of the form (6), we choose a resolution of the identity $(e_j), j=1,2,\ldots,$ such that $e_j \in K_\infty$, $\operatorname{supp} e_j \subset (t_j-a, t_j+a)$ for some $t_j \in R^1$, $j=1,2,\ldots,$ and $\sum_j e_j(s)=1$ $(s\in R^1)$. Writing $\varphi=\sum_j \varphi e_j, \psi=\sum_j \psi e_j$ and applying the considerations of the last paragraph to the functions $\varphi e_j, \psi e_j$, we define a scalar product on K_∞ by

$$[\varphi, \psi] := [\sum_j \tilde{U}_{t_j}\tilde{V}_{-t_j}(\varphi e_j), \sum_j \tilde{U}_{t_j}\tilde{V}_{-t_j}(\psi e_j)]_K.$$

It is not hard to show that this is a continuous bilinear functional on K_∞ which is invariant under translation. Therefore, according to [9, II, § 3.5], it is of the form $(\tilde{F}, \varphi^\circ * \bar{\psi})$ with some generalized function $\tilde{F} \in \mathfrak{P}^d_{\varkappa;\infty}$ which extends F to the real axis. The proposition is proved.

Remark. It can be shown that there is a one-to-one correspondence between all continuations $\tilde{F} \in \mathfrak{P}^d_{\varkappa;\infty}$ of $F \in \mathfrak{P}^d_{\varkappa;a}$ to the whole real axis and all generalized resolvents of the operator A_0, cf. [6, Part IV].

3. The generalized functions $F \in \mathfrak{P}^d_{\varkappa;a}$ are "conditionally positive definite" in the following sense *): There exists a polynomial p of degree $\leqq \varkappa$ such that

$$\left(p\left(i\frac{d}{dt}\right)\bar{p}\left(i\frac{d}{dt}\right)F, \varphi^\circ * \bar{\varphi}\right) \geqq 0 \quad (\varphi \in K_a), \tag{7}$$

cf. [10]. If p is chosen monic (that is, the coefficient of the term with greatest exponent is 1) and of minimal possible degree, it is unique if and only if the operator A_0 is π-self-adjoint in $\Pi^K_\varkappa(F)$, cf. [6, Part IV, § 2].

The generalized function $\tilde{F} \in \mathfrak{P}^d_{\varkappa;\infty}$ admits an (essentially unique) integral representation by means of a "spectral measure" μ of exponential growth at infinity, see [10]. This representation has the same structure as that appearing in [9, II, § 4, Theorem 3]. We decompose the integral into the sum of two integrals. One of them is taken over a bounded interval containing all singularities of the "spectral measure" μ (in [9, II, § 4, (25)] the only such singularity of μ is at zero, while in general the singularities may appear at eigenvalues of A_0 with nonpositive eigenvectors). The second integral without the "regularizing term" defines a positive definite generalized function, whereas the first integral (over the bounded interval) and those terms which are given by the nonreal spectrum correspond to a generalized function induced by a continuous function. Thus the following proposition has been proved.

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*) In [9] the generalized function F on R^1 is called "conditionally positive definite" if (7) holds with a homogeneous polynomial p.

Proposition 2. *The generalized function* $\tilde{F}\in\mathfrak{P}^d_{\varkappa;\infty}$ *can be decomposed as* $\tilde{F}=\tilde{F}_1+\tilde{F}_0$, *where* $\tilde{F}_1\in\mathfrak{P}_{\varkappa;\infty}$ *is a continuous function and* $\tilde{F}_0\in\mathfrak{P}^d_{0;\infty}$ *is a positive definite generalized function.*

Combining Propositions 1 and 2 we obtain:

Corollary. *The generalized function* $F\in\mathfrak{P}^d_{\varkappa;a}$ *can be decomposed as* $F=F_1+F_0$, *where* $F_1\in\mathfrak{P}_{\varkappa;a}$ *and* $F_0\in\mathfrak{P}^d_{0;a}$.

4. Proof of the Theorem

1. We start with the following lemma (cf. [11, IX. 2]).

Lemma 2. *Let* f_0 *be a function on* $(-2a, 2a)$ *which is locally bounded, measurable and positive definite as a generalized function.**) Then it admits a representation*

$$f_0(t) = \int_{R^1} e^{i\lambda t}\, d\mu_0(\lambda) \quad \text{for almost all} \quad t\in(-2a, 2a) \tag{8}$$

with a bounded nonnegative measure μ_0 *on* R^1.

Proof. The positive definite generalized function f_0 has an extension $\tilde{f}_0\in\mathfrak{P}^d_{0;\infty}$, see Proposition 1. Then $\tilde{f}_0$ is the Fourier transform of a nonnegative polynomially bounded measure μ_0 on R^1. In particular,

$$(\tilde{f}_0, \varphi) = \int_{-\infty}^{\infty} \phi(\lambda)\, d\mu_0(\lambda) \quad (\varphi\in K_a),$$

where $\phi(\lambda) := \int_{-a}^{a} e^{i\lambda t}\varphi(t)\,dt \quad (\lambda\in R^1)$.

Let $j\in K_\infty$ be such that $j\geqq 0$, $\int_{-a}^{a} j(t)\,dt=1$, $\operatorname{supp} j\subset(-a, a)$ and for $0<\varepsilon\leqq 1$ define the functions

$$j_\varepsilon(t) := \varepsilon^{-1}\, j(t\varepsilon^{-1}) \quad (t\in R^1).$$

Then

$$\int_{-\infty}^{\infty} |\hat{j}_\varepsilon(\lambda)|^2\, d\mu_0(\lambda) = (\tilde{f}_0, j_\varepsilon * j_\varepsilon) \leqq \sup_{t\in \operatorname{supp} j} |f_0(2t)|.$$

On each compact subset of R^1 the functions $\hat{j}_\varepsilon$ tend uniformly to 1 if $\varepsilon\downarrow 0$. Hence

$$\int_{-\infty}^{\infty} d\mu_0(\lambda) \leqq \sup_{t\in \operatorname{supp} j} |f_0(2t)|,$$

**) In [11, IX, 2] a bounded function on R^1 with this property is called *weakly positive definite.*

and, since j can be chosen to have arbitrarily small support, it follows that

$$\int_{-\infty}^{\infty} d\mu_0(\lambda) \leqq \varlimsup_{t \downarrow 0} |f_0(t)|.$$

The lemma is proved.

2. Now let $f \in \mathfrak{P}^m_{\varkappa;\, a}$. Then f can be considered as a generalized function, and since we have

$$[\varphi, \psi]_K = (f, \varphi^\circ * \bar{\psi}) = \int_{-a}^{a} \int_{-a}^{a} f(t-s)\varphi(s)\overline{\psi(t)}\, ds\, dt,$$

according to the results of section 2.2 this scalar product has $\varkappa'$, $0 \leqq \varkappa' \leqq \varkappa$, negative squares. By the Corollary to Proposition 2 the generalized function f can be decomposed as

$$f = f_1 + f_0, \tag{9}$$

where $f_1 \in \mathfrak{P}_{\varkappa';\, a}$, $f_0 \in \mathfrak{P}^d_{0;\, a}$, and the equality holds in the sense of generalized functions. Evidently, $f_0 = f - f_1$ can be considered as a locally bounded measurable function. Thus, by Lemma 2, it admits a representation (8) with some bounded measure μ_0. The continuous function

$$f_c(t) := f_1(t) + \int_{-\infty}^{\infty} e^{i\lambda t}\, d\mu_0(\lambda) \quad (|t| < 2a) \tag{10}$$

belongs to some class $\mathfrak{P}_{\varkappa'';a}$, $0 \leqq \varkappa'' \leqq \varkappa$, and the relations (8), (9) and (10) imply

$$f(t) = f_c(t) + f_s(t) \quad (|t| < 2a),$$

where $f_s(t) = 0$ a.e. on $(-2a, 2a)$. We show that the function f_s is positive definite. To this end we first prove the following

Lemma 3. *Let g be a complex function on $(-2a, 2a)$ such that $g(t) = \overline{g(-t)}$ and $g(t) = 0$ a.e. on $(-2a, 2a)$. On the linear set $\mathscr{L}_0$* (see section 2.1) *we consider the scalar product*

$$[u, v] := \sum_{-a < s,\, t < a} g(t-s)u(s)\overline{v(t)} \quad (u, v \in \mathscr{L}_0).$$

If there exists a $u_0 \in \mathscr{L}_0$ such that $[u_0, u_0] < 0$ then we can find a set $L_s \subset \mathscr{L}_0$ with the following properties:

a) $\operatorname{card} L_s > \aleph_0$,

b) $[u, u] = -1$, $[u, v] = 0$ *if* $u, v \in L_s$, $u \neq v$.

Proof. Let $u_0 = \sum_{j=1}^{n} \alpha_j \varepsilon_{t_j}$, $[u_0, u_0] = -1$. We choose $\delta > 0$ so that $t_j \pm \delta \in (-a, a)$

for $j=1, 2, \dots, n$. Then

$$[V_\sigma u_0, V_\sigma u_0] = [u_0, u_0] = -1 \quad \text{for all } |\sigma| < \delta.$$

The set $\Delta_0 := \{s: |s| < 2a, g(s) = 0\}$ has Lebesgue measure $\lambda(\Delta_0) = 2a$.

We denote by G the family of all nonempty sets Γ such that $\Gamma \subset (-\delta, \delta)$, and the relations $\sigma, \tau \in \Gamma, \sigma \neq \tau$ imply $\sigma - \tau + (t_j - t_k) \in \Delta_0$ for $j, k = 1, 2, \dots, n$. Then G is not empty. Indeed, for arbitrary σ, $|\sigma| < \delta$, define

$$\Delta_{jk}(\sigma) := (\sigma + (t_j - t_k) - \Delta_0) \cap (-\delta, \delta), \quad j, k = 1, 2, \dots, n.$$

Then $\lambda(\Delta_{jk}(\sigma)) = 2\delta$, which implies $\lambda\left(\bigcap_{j,k=1}^{n} \Delta_{jk}(\sigma)\right) = 2\delta$, hence $\bigcap_{j,k=1}^{n} \Delta_{jk}(\sigma) \neq \emptyset$. For an arbitrary $\tau \neq \sigma$ which belongs to this intersection we have $\{\sigma, \tau\} \in G$.

The family G is partially ordered by inclusion, and each of its totally ordered subfamilies has an upper bound. We show that the maximal elements of G are not countable. Indeed, assume that a maximal element $\Gamma_{\max}$ of G is countable: $\Gamma_{\max} = \{s_\nu : \nu = 1, 2, \dots\}$. Consider the set $\Delta := \bigcap_{j,k,\nu} \Delta_{jk}(s_\nu)$. Then we have again $\lambda(\Delta) = 2\delta$, and therefore $\Delta \setminus \Gamma_{\max} \neq \emptyset$. If $\tau \in \Delta \setminus \Gamma_{\max}$, then $\Gamma_{\max} \cup \{\tau\} \in G$, which contradicts the maximality of $\Gamma_{\max}$.

Now let $\Gamma_0 \in G$, $\operatorname{card} \Gamma_0 > \aleph_0$ and put $L_s := \{V_\sigma u_0 : \sigma \in \Gamma_0\}$. Then we have

$$[V_\sigma u_0, V_\sigma u_0] = [u_0, u_0] = -1,$$

$$[V_\tau u_0, V_\sigma u_0] = \sum_{j,k=1}^{n} \alpha_j \overline{\alpha_k} g(\sigma - \tau + t_j - t_k) = 0 \quad \text{if} \quad \sigma \neq \tau.$$

The lemma is proved.

Now we show that f_s is positive definite. Assuming the contrary we find a subset L_s of $\mathcal{L}_0$ with the properties a), b) of Lemma 3 for $g = f_s$. Denote the elements of L_s by $u_\gamma, \gamma \in \Gamma_0$. The space $\Pi_{\varkappa''}(f_c)$ is separable. Hence there exists a countable subset Γ_1 of Γ_0 such that the elements $u_\gamma, \gamma \in \Gamma_1$, form a total set in the subspace of $\Pi_{\varkappa''}(f_c)$ generated by $u_\gamma, \gamma \in \Gamma_0$. Choose $n > \varkappa$, and mutually different elements $u^1, \dots, u^n \in L_s$ which do not belong to $\{u_\gamma : \gamma \in \Gamma_1\}$. Then, if $\|\cdot\|_c$ denotes a Hilbert norm on $\Pi_{\varkappa''}(f_c)$ which corresponds to some fundamental decomposition, then to each u^j there exists a finite sum $\sum_{\gamma \in \Gamma_1} \xi_\gamma^{(j)} u_\gamma$ such that for $y^j := u^j - \sum_{\gamma \in \Gamma_1} \xi_\gamma^{(j)} u_\gamma$ we have

$$\|y^j\|_c^2 < \frac{1}{2n^2}, \quad j = 1, 2, \dots, n.$$

On the other hand, denoting by $[\cdot, \cdot]_s$ the scalar product on $\mathcal{L}_0$ corresponding to f_s, we find

$$[y^j, y^k]_s = -\delta_{jk} - \sum_{\gamma \in \Gamma_1} \xi_\gamma^{(j)} \overline{\xi_\gamma^{(k)}}, \quad j, k = 1, 2, \dots, n.$$

Hence, for arbitrary complex numbers $\eta_1, \ldots, \eta_n$ it follows that

$$\|\eta_j y^j\|_c^2 \leqq \left(\sum_{j=1}^n |\eta_j| \|y^j\|_c\right)^2 \leqq \frac{1}{2n} \sum_{j=1}^n |\eta_j|^2,$$

$$\left[\sum_{j=1}^n \eta_j y^j, \sum_{j=1}^n \eta_j y^j\right]_s = -\sum_{j=1}^n |\eta_j|^2 - \sum_{\gamma \in \Gamma_1} \left|\sum_{j=1}^n \xi_\gamma^{(j)} \eta_j\right|^2 \leqq -\sum_{j=1}^n |\eta_j|^2,$$

and we get finally

$$\left[\sum_{j=1}^n \eta_j y^j, \sum_{j=1}^n \eta_j y^j\right] \leqq \sum_{j=1}^n |\eta_j|^2 \left(-1 + \frac{1}{2n}\right).$$

However, this is impossible, since the scalar product on the left hand side has at most $\varkappa$ negative squares on $\mathscr{L}_0$. This contradiction implies that f_s is positive definite, and the Theorem is proved.

3. The decomposition (1) can be written in a more geometric form. To this end we first observe that in (3) the right hand side can be replaced by

$$\int_{-a}^{a} \int_{-a}^{a} f_c(t-s) \varphi(s) \overline{\psi(t)} \, ds \, dt,$$

and the space $\Pi^C(f)$ can be identified with $\Pi_\varkappa(f_c)$. Therefore it is also a $\pi_\varkappa$-space and we shall write $\Pi_\varkappa^C(f)$ instead of $\Pi^C(f)$. As a nondegenerate subspace of $\Pi_\varkappa(f)$, it is the range of a π-orthogonal projector P in $\Pi_\varkappa(f)$, and we have a decomposition

$$\Pi_\varkappa(f) = \Pi_\varkappa^C(f) \oplus \Pi_0(f),$$

where $\Pi_0(f)$ is a Hilbert space with respect to the scalar product $[\cdot, \cdot]$.

Further, if $\varphi \in C_a$, then (4) yields

$$[\varepsilon_s, \varphi] = \int_{-a}^{a} f(t-s) \overline{\varphi(t)} \, dt = \int_{-a}^{a} f_c(t-s) \overline{\varphi(t)} \, dt \quad (|s| < a),$$

and if $(\delta_t^{(n)})$, $n = 1, 2, \ldots$, is a δ_t-sequence of elements of C_a, then we find

(11) $$[\varepsilon_s, \delta_t^{(n)}] \to f_c(t-s) \quad (n \to \infty;\ |s|, |t| < a).$$

Moreover, for arbitrary $\psi \in C_a$ we have

$$[\psi, \delta_t^{(n)}] \to [\psi, \varepsilon_t] \quad (n \to \infty).$$

This relation implies $\delta_t^{(n)} \to P\varepsilon_t$ $(n \to \infty)$ in the weak topology of $\Pi_\varkappa^C(f)$ or $\Pi_\varkappa(f)$, and from (11) we get finally that

$$[\varepsilon_s, P\varepsilon_t] = f_c(t-s) \quad (|s|, |t| < a).$$

Thus the decomposition (1) can be written as

$$f(2t) = [P\varepsilon_{-t}, \varepsilon_t] + [(I-P)\varepsilon_{-t}, \varepsilon_t] \quad (|t| < a).$$

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